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ORIGINAL PAPER

Dynamic analysis of Michaelis–Menten chemostat-type competition models with time delay and pulse in a polluted environment

Xinzhu Meng · Zhenqing Li · Juan J. Nieto

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Abstract In this paper, a new Michaelis–Menten type chemostat model with time delay and pulsed input nutrient concentration in a polluted environment is considered. We obtain a 'microorganism-extinction' semi-trivial periodic solution and establish the sufficient conditions for the global attractivity of the semi-trivial periodic solution. By use of new computational techniques for impulsive differential equations with delay, we prove and support with numerical calculations that the system is permanent. Our results show that time delays and the polluted environment can lead the microorganism species to be extinct.

Keywords Permanence · Impulsive input · Michaelis–Menten type chemostat model · Time delay for growth response · Extinction

1 Introduction

A chemostat is basically a culture vessel having an input aperture for the influx of sterile nutrient medium from a reservoir and an overflow aperture for the efflux of exhausted medium, living cells, and cellular debris. The device (and the term "chemo-

X. Meng
Information School, Shandong University of Science and Technology, Qingdao 266510, People's Republic of China

X. Meng · Z. Li (✉)
State Key Laboratory of Vegetation and Environmental Change, Institute of Botany, Chinese Academy of Sciences, Beijing 100093, People's Republic of China
e-mail: lizq@ibcas.ac.cn; mxz0011@yahoo.cn

J. J. Nieto
Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Campostela, Spain
e-mail: juanjose.nieto.roig@usc.es

stat") was invented by Novick and Szilard [1]. Chemostats consist essentially of two primary parts: a sterile nutrient reservoir and a growth chamber. By controlling the levels of the limiting nutrient, bacterial culture in the growth chamber can be kept at a reduced growth rate over an indefinite period of time [2]. As medium flows out from the growth chamber, bacteria and byproducts can be harvested. For example, chemostats are frequently used in the industrial manufacturing of ethanol. Chemostats also allow this yield and growth rate to be controlled independently. A biomathematical model of the process would be helpful in developing strategies and since commercial production can take place on a scale of many generations, it is important to understand the asymptotic behavior of the system. Chemostats with period inputs are studied in [3], with periodic washout rate in [4] and with periodic input and washout in [5]. However, existing theories on chemostat models largely ignore the effects of environmental pollution. Environmental pollution caused by various industries and pesticides used in agriculture is one of the most important social and ecological problems at present. In order to use and regulate toxic substance wisely, we must assess the risk of the population exposed to toxicant. Therefore, it is important to study the effects of toxicant on populations and to find a theoretical threshold value, which determines permanence or extinction of a population community. Many mathematical chemostat models have been considered in the literature [3–17]. While the Monod model [7] has some success in describing steady state growth rates, it has been found inadequate to predict transients observed in chemostat experiments where the initial data is not at the globally attracting steady state. Lag phases occur in the growth response of microorganisms to changes in the environment and the models take the form of delay differential equations [8–15]. We refer to Hsu and Tseng [10], Wolkowicz and Xia [11], and the references therein, for more detailed discussions on chemostat modeling approaches using delay differential equations [8–15]. The microbial continuous culture has been investigated in [1–17] and some interesting results were obtained. Many scholars pointed out that it was necessary and important to consider models with periodic perturbations, since these models might be quite naturally exposed in many real world phenomena (for instance, food supply, mating habits, harvesting). Systems with sudden perturbations lead to impulsive differential equations, which have been studied intensively and systematically in [18–28]. The authors in [29–32] introduced some impulsive differential equations in population dynamics and exhibited complex behavior of impulsive equations. In recent years, the research on the chemostat model with impulsive perturbations is a relevant subject in mathematical biology, but not totally developed (see [33–37] and the references therein). Furthermore, high dimensional delay chemostat models with pulse have never been seen by now. However, this is an interesting problem in mathematical biology.

Therefore, it is relevant to introduce delayed growth response, impulsive input nutrient concentration and impulsive input toxicant concentration to a chemostat model. While delay differential equations have been widely used in modeling population dynamics, some practical problems have to be overcome when applied to models of the chemostat. We remark that the dynamics of impulsive and delayed differential equations are usually more difficult to study than that of ordinary differential equations. As a result, fewer analytic tools are available for studying the dynamics of impulsive and delayed differential equation, so, chemostat models with impulse and delay are not extensive. In this paper, we consider a Michaelis–Menten type competition chemostat model with impulsive input nutrient concentration and delayed growth response in a polluted environment, and investigate how the impulsive perturbation of the substrate, time delay for growth response and impulsive input toxicant affect the dynamic behavior of the chemostat system.

2 Model and preliminaries

Hsu and coworkers [9,10] studied a chemostat model with the microbial continuous culture

$$\begin{aligned} S'(t) &= D(S_0 - S(t)) - \sum_{i=1}^{2} \frac{\mu_i S(t) x_i(t)}{\delta_i (K_i + S(t))}, \\ x_1'(t) &= \frac{\mu_1 S(t) x_1(t)}{K_1 + S(t)} - D x_1(t), \\ x_2'(t) &= \frac{\mu_2 S(t) x_2(t)}{K_2 + S(t)} - D x_2(t), \end{aligned} \tag{1}$$

where $S(t)$ denotes the concentration of the unconsumed nutrient in the growth vessel at time t and $x_i(t)(i = 1, 2)$ denote the biomass of two populations of microorganisms at time t. S_0 and D are positive constants and denote, respectively, the concentration of the growth-limiting nutrient and the flow rate of the chemostat. The function $p(S) = \dfrac{\mu_i S(t) x_i(t)}{\delta_i (K_i + S(t))}$ represents two species specific per-capita nutrient uptake rate. More details can be seen in the papers of Hsu and coworkers [9,10]. Hsu and coworkers [9,10] considered chemostat-type competition models with continuous culture and presented some basic progress on global qualitative analysis of solution to (1).

The taken nutrient cannot translate instantaneously into viable microorganisms, that is, there is a time delay in the growth response that describes the lag involved in the nutrient conversion process. At the same time the pulsed input concentration the toxicant may lead to the microorganisms species be extinct in the polluted chemostat environment. Therefore, we consider the polluted Michaelis–Menten type chemostat model with pulsed input and delayed growth response in this paper.

The goal of this paper is to give a description of some of the basic dynamical properties of a chemostat model with delayed growth response and impulsive perturbation on the nutrient concentration in a polluted environment, which incorporates the Michaelis–Menten functional response and two competitive predator species x_1, x_2. The model takes the form

$$\begin{aligned} S'(t) &= -DS(t) - \sum_{i=1}^{2} \frac{\mu_i S(t) x_i(t)}{\delta_i (K_i + S(t))}, \quad t \neq nT,\ n \in N, \\ x_1'(t) &= e^{-D\tau_1} \frac{\mu_1 S(t-\tau_1) x_1(t-\tau_1)}{K_1 + S(t-\tau_1)} \\ &\quad -\beta_1 x_2 x_1 - (D + r_1 c(t)) x_1(t), \quad t \neq nT,\ n \in N, \end{aligned}$$

$$
\begin{aligned}
x_2'(t) &= e^{-D\tau_2}\frac{\mu_2 S(t-\tau_2)x_2(t-\tau_2)}{K_2+S(t-\tau_2)} \\
&\quad -\beta_2 x_1 x_2 - (D + r_2 c(t))x_2(t), \quad t \neq nT,\ n \in N, \\
c'(t) &= -Dc(t), \quad t \neq nT,\ n \in N, \\
\Delta S &= S(nT^+) - S(nT) = \gamma S_0,\ \Delta x_1 = x_1(nT^+) - x_1(nT) = 0,\ t = nT,\ n \in N, \\
\Delta x_2 &= x_2(nT^+) - x_2(nT) = 0,\ \Delta c = c(nT^+) - c(nT) = \gamma c_0,\ t = nT,\ n \in N, \\
S(0^+) &\geq 0,\ x_1(0^+) \geq 0,\ x_2(0^+) \geq 0,\ c(0^+) \geq 0
\end{aligned} \tag{2}
$$

where $c(t)$ is the concentration of the toxicant in the chemostat, $r_i > 0 (i = 1, 2)$ is the rate of death of the killed microorganisms by the toxicant. $\Delta S = T = \gamma/D$ is the period of the pulsing, γS_0 is the amount of limiting substrate pulsed each T. DS_0 units of substrate are added, on average, per unit of time. γc_0 is the amount of pulsed input concentration the toxicant at each T. The constant $\tau_i \geq 0 (i = 1, 2)$ denotes the time delay involved in the conversion of nutrient to viable biomass. The positive constant, $e^{-D\tau_i}(i = 1, 2)$, is required, because it is assumed that the current change in biomass depends on the amount of nutrient consumed $\tau_i(i = 1, 2)$ units of time in the past by the microorganisms that were in the growth vessel at that time and managed to remain in the growth vessel the $\tau_i(i = 1, 2)$ units of time required to process the nutrient. $S(nT^+) = \lim_{t\to nT^+} S(t)$, and $S(t)$ is left continuous at $t = nT$, i.e., $S(nT) = \lim_{t\to nT^-} S(t), x_i(t)(i = 1, 2)$ is continuous for all $t \geq 0$, $c(nT^+) = \lim_{t\to nT^+} c(t)$, and $c(t)$ is left continuous at $t = nT$, i.e., $c(nT) = \lim_{t\to nT^-} c(t)$, the details can be seen in the books of Laksmikantham et al. [18], Haddad et al. [19] and Zavalishchin and Sesekin [20].

Motivated by the application of systems (2) to population dynamics (refer to [38]), we assume that solutions of systems (2) satisfy

$$S, x_1, x_2, c \in C_+. \tag{3}$$

Lemma 2.1 (see [18]) *Consider the following impulse differential inequalities:*

$$
\begin{aligned}
&w'(t) \leq (\geq) p(t)w(t) + q(t), \quad t \neq t_k, \\
&w(t_k^+) \leq (\geq) d_k w(t_k) + b_k, \quad t = t_k,\ k \in N,
\end{aligned}
$$

where $p, q \in C(R_+, R)$, $d_k \geq 0$, and b_k are constants.

Assume

- (A_0) *the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots$, with $\lim_{t\to\infty} t_k = \infty$;*
- (A_1) *$w \in PC'(R_+, R)$ and $w(t)$ is left-continuous at t_k, $k \in N$.*

Then

$$w(t) \le (\ge) w(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\theta)d\theta\right) q(s)ds, \quad t \ge t_0. \tag{4}$$

Lemma 2.2 ([38]) *Consider the following delay differential equation*

$$\frac{dx(t)}{dt} = ax(t-\tau) - bx(t),$$

where a, b, τ *are all positive constants and* $x(t) > 0$ *for* $t \in [-\tau, 0]$.

(i) *If* $a < b$, *then* $\lim_{t\to\infty} x(t) = 0$.
(ii) *If* $a > b$, *then* $\lim_{t\to\infty} x(t) = +\infty$.

For convenience, we give the basic properties of the following system:

$$\begin{aligned} S'(t) &= -DS(t), \quad t \neq nT, \ n \in N, \\ S(t^+) &= S(t) + \gamma S_0, \quad t = nT, \ n \in N, \\ S(0^+) &= S_{10} \ge 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} c'(t) &= -Dc(t), \quad t \neq nT, \ n \in N, \\ c(t^+) &= c(t) + \gamma c_0, \quad t = nT, \ n \in N, \\ c(0^+) &= c_{10} \ge 0. \end{aligned} \tag{6}$$

Lemma 2.3 *System* (5) *has a positive periodic solution* $S^*(t)$ *and for any solution* $S(t)$ *of* (5) *with initial value* $S_{10} \ge 0$, *we get* $|S(t) - S^*(t)| \to 0$ *as* $t \to \infty$, *moreover*

(i) *if* $S_{10} \ge \frac{\gamma S_0}{1-e^{-DT}}$, *then* $S(t) \ge S^*(t)$;
(ii) *if* $S_{10} < \frac{\gamma S_0}{1-e^{-DT}}$, *then* $S(t) < S^*(t)$

where $S^*(t) = \frac{\gamma S_0 e^{-D(t-nT)}}{1-e^{-DT}}, t \in (nT, (n+1)T], n \in N, S^*(0^+) = \frac{\gamma S_0}{1-e^{-DT}}$.

Proof For integrating and solving the first equation of system (5) between pulses, yields

$$S(t) = S(nT)e^{-D(t-nT)}, \quad nT < t \le (n+1)T,$$

where $S(nT)$ be the initial value at time nT. Using the second equation of system (5), we deduce that

$$S((n+1)T) = S(nT)e^{-DT} + \gamma S_0 = f(S(nT)), \tag{7}$$

where $f(S) = e^{-\mu T} S + \gamma S_0$. Equation 7 has a unique positive equilibrium $S^* = \frac{\gamma S_0}{1-e^{-DT}}$. Since $f(S)$ is a straight line with respect to S with slope less than 1, we obtain that S^* is globally asymptotically stable. This implies that the corresponding periodic solution of system (5), is globally asymptotically stable. It is clear that

$$S^*(t) = \frac{\gamma S_0 e^{-D(t-nT)}}{1-e^{-DT}}, \quad t \in (nT, (n+1)T],\ n \in N, \quad S^*(0^+) = \frac{\gamma S_0}{1-e^{-DT}}$$

is a positive periodic solution of (5). The solution of (5) is $S(t) = (S(0^+) - S^*(0^+))\, e^{-Dt} + S^*(t)$, $t \in (nT, (n+1)T]$, $n \in N$. Hence $|S(t) - S^*(t)| \to 0$ as $t \to \infty$. And $S(t) \geq S^*(t)$ if $S_{10} \geq \frac{\gamma S_0}{1-e^{-DT}}$ and $S(t) < S^*(t)$ if $S_{10} < \frac{\gamma S_0}{1-e^{-DT}}$. This proof is complete. □

Lemma 2.4 *System (6) has a positive periodic solution $c^*(t)$ and for any solution $c(t)$ of (6) with initial value $c_{10} \geq 0$, we get $|c(t) - c^*(t)| \to 0$ as $t \to \infty$, moreover*

(i) *if $c_{10} \geq \frac{\gamma c_0}{1-e^{-DT}}$, then $c(t) \geq c^*(t)$;*
(ii) *if $c_{10} < \frac{\gamma c_0}{1-e^{-DT}}$, then $c(t) < c^*(t)$*

where $c^*(t) = \frac{\gamma c_0 e^{-D(t-nT)}}{1-e^{-DT}}$, $t \in (nT, (n+1)T]$, $n \in N$, $c^*(0^+) = \frac{\gamma c_0}{1-e^{-DT}}$.

Lemma 2.4 can be analyzed by the same method as the above Lemma 2.3. So we omit it.

Lemma 2.5 *If $(S(t), x_1(t), x_2(t), c(t))$ is any solution of system (2) with initial condition (3), then there exists any small constant $\epsilon > 0$ such that $S(t) \leq \frac{\gamma S_0}{1-e^{-DT}} + \epsilon =: \eta$, $x_i(t) \leq \delta_i \gamma S_0 \frac{e^{-D\tau} e^{DT}}{e^{DT}-1} + \epsilon =: L_i$, $i = 1, 2$ and $0 < m_4 \leq c(t) \leq M_4$ where $\tau = \max\{\tau_1, \tau_2\}$, $m_4 = \frac{\gamma c_0 e^{-DT}}{1-e^{-DT}} - \epsilon$ and $M_4 = \frac{\gamma c_0}{1-e^{-DT}} + \epsilon$ for t large enough.*

Proof Let $(S(t), x_1(t), x_2(t), c(t))$ be any solution of system (2) with initial condition (3).

Let

$$W(t) = e^{-D\tau} S(t) + \sum_{i=1}^{2} \frac{1}{\delta_i} x_i(t + \tau_i).$$

The upper right derivative of $W(t)$ along the trajectories of (2) is

$$\begin{aligned}\dot{W}(t) &\leq -De^{-D\tau} S(t) - \frac{D}{\delta_1} x_1(t+\tau_1) - \frac{D}{\delta_2} x_2(t+\tau_2)\\ &= -DW(t)\end{aligned}$$

Consider the following impulse differential inequalities:

$$W'(t) \le -DW(t), \quad t \ne nT, n \in N,$$
$$W(t^+) = W(t) + e^{-D\tau}\gamma S_0, \quad t = nT, \ n \in N,$$

by Lemma 2.1, we obtain

$$W(t) \le W(0^+)e^{-Dt} + e^{-D\tau}\gamma S_0 \frac{e^{-D(t-T)}}{1-e^{-DT}}$$
$$+ e^{-D\tau}\gamma S_0 \frac{e^{DT}}{e^{DT}-1} \to e^{-D\tau}\gamma S_0 \frac{e^{DT}}{e^{DT}-1}, \ t \to \infty.$$

According to the definition of $W(t)$, it can be seen that $S(t) \le \eta$ and $x_i(t) \le L_i$ $(i = 1, 2)$ for t large enough. From system (6), we have that

$$c^*(t) = \frac{\gamma c_0 e^{-D(t-nT)}}{1-e^{-DT}}, \quad t \in (nT, (n+1)T], \ n \in N, \ c^*(0^+) = \frac{\gamma c_0}{1-e^{-DT}}$$

is a globally asymptotically stable positive periodic solution of system (6). Hence, we have that

$$\frac{\gamma c_0 e^{-DT}}{1-e^{-DT}} \le c^*(t) \le \frac{\gamma c_0}{1-e^{-DT}}, \quad t \ge 0.$$

By Lemma 2.4, we get that

$$0 < m_4 = \frac{\gamma c_0 e^{-DT}}{1-e^{-DT}} - \epsilon \le c(t) \le M_4 = \frac{\gamma c_0}{1-e^{-DT}} + \epsilon$$

for ϵ small enough and t large enough. The proof is complete. □

Let $R_+ = [0, +\infty)$, $R_+^4 = \{X \in R^4 : X \ge 0, X = (S, x_1, x_2, c)\}$, N be the set of nonnegative integers. Denote $f = (f_1, f_2, f_3, f_4)^T$ the map defined by the right-hand of the anterior two equations of system (2). Let $V : R_+ \times R_+^4 \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(nT, (n+1)T] \times R_+^4$ and for each $X \in R_+^4, n \in N$, $\lim_{(t,y)\to((nT)^+,X)} V(t, y) = V((nT)^+, X)$ exists;
(ii) V is locally Lipschitzian in X.

Lemma 2.6 ([18]) *Let $V : R_+ \times R_+^4 \to R_+$, and $V \in V_0$. Assume that*

$$D^+V(t, z(t)) \le (\ge) g(t, V(t, z(t))), \quad t \ne nT,$$
$$V(t, z(t)^+) \le (\ge) \Psi_n(V(t, z(t))), \quad t = nT,$$

where $g : R_+ \times R_+ \to R$ is continuous in $(nT, (n+1)T] \times R_+$ and for each $x \in R_+, n \in N$, $\lim_{(t,y)\to((nT)^+,x)} g(t, y) = g((nT)^+, x)$ exist; $\Psi_n : R_+ \to R_+$ is *nondecreasing. Let* $r(t) = r(t, 0, u_0)$ *be the maximal (minimal) solution of the scalar impulsive differential equation*

$$u' = g(t, u), \quad t \neq nT,$$
$$u(t^+) = \Psi_n(u(t)), \quad t = nT,$$

existing on $[0, \infty)$. *Then* $V(0^+, z_0) \leq (\geq) u_0$ *implies that* $V(t, z(t)) \leq (\geq) r(t)$, $t \geq 0$, *where* $z(t) = z(t, 0, z_0)$ *is any solution of* (2) *existing on* $[0, \infty)$.

3 The main results

First, we investigate the extinction of the microorganism species, that is, microorganism are entirely absent from the chemostat permanently, i.e.,

$$x_1(t) = x_2(t) = 0, \quad t \geq 0.$$

This is motivated by the fact that $x_i^* = 0 (i = 1, 2)$ is an equilibrium solution for the variable $x_i(t) (i = 1, 2)$, as it leaves $x_i'(t) = 0 (i = 1, 2)$. Under these conditions, we show below that the nutrient concentration oscillates with period T in synchronization with the periodic impulsive input nutrient concentration, and the concentration of the toxicant in the chemostat oscillates with period T in synchronization with the periodic impulsive input the toxicant.

By Lemma 2.5 and the second and the third equations of system (2), it follows that

$$x_i'(t) \leq \mu_i e^{-D\tau_i} x_i(t - \tau_i) - (D + r_i m_4) x_i(t), \quad i = 1, 2$$

where $m_4 = \dfrac{\gamma c_0 e^{-DT}}{1 - e^{-DT}} - \epsilon$. Clearly, if $\mu_i e^{-D\tau_i} < D + r_i \dfrac{\gamma c_0 e^{-DT}}{1 - e^{-DT}}$, then $\mu_i e^{-D\tau_i} < D + r_i \dfrac{\gamma c_0 e^{-DT}}{1 - e^{-DT}} - \epsilon$ for ϵ small enough. By Lemma 2.2, we have $\lim_{t\to\infty} x_i(t) = 0$ if $\mu_i e^{-D\tau_i} < D + r_i \dfrac{\gamma c_0 e^{-DT}}{1 - e^{-DT}}$, which implies the microorganism species becomes ultimately extinct. This shows that the specific growth of the microorganism species can not supply the losing of the microorganism species to flow out and the death of the microorganism species which is killed by toxicant no matter how much input the nutrient. Therefore, we assume $\mu_i e^{-D\tau_i} > D + r_i \frac{\gamma c_0 e^{-DT}}{1 - e^{-DT}} (i = 1, 2)$ in the rest of this paper.

By Lemmas 2.3 and 2.4, we have the following Lemma 3.1.

Lemma 3.1 *Systems* (5) *and* (6) *have a unique positive periodic solution* $S^*(t)$ *and* $c^*(t)$, *respectively, that is, the system* (2) *has a 'microorganism-extinction' semi-trivial periodic solution* $(S^*(t), 0, 0, c^*(t))$ *for* $t \in (nT, (n + 1)T]$, $n \in N$, *for any solution* $(S(t), x_1(t), x_2(t), c(t))$ *of* (1) *we have* $S(t) \to S^*(t)$ *and* $c(t) \to c^*(t)$ *as* $t \to \infty$.

Theorem 3.1 *Periodic solution* $(S^*(t), 0, 0, c^*(t))$ *of system (2) is globally attractive if*

$$\gamma S_0 < \min\left\{\frac{K_1\left(1-e^{-DT}\right)\left[D\left(e^{DT}-1\right)+r_1\gamma c_0\right]}{\left(\mu_1 e^{-D\tau_1}-D\right)\left(e^{DT}-1\right)-r_1\gamma c_0},\right.$$
$$\left.\frac{K_2\left(1-e^{-DT}\right)\left[D\left(e^{DT}-1\right)+r_2\gamma c_0\right]}{\left(\mu_2 e^{-D\tau_2}-D\right)\left(e^{DT}-1\right)-r_2\gamma c_0}\right\} \tag{8}$$

or

$$\gamma c_0 > \max\left\{\frac{\left(e^{DT}-1\right)\left[\gamma S_0\left(\mu_1 e^{-D\tau_1}-D\right)-K_1 D\left(1-e^{-DT}\right)\right]}{r_1\left[\gamma S_0+K_1\left(1-e^{-DT}\right)\right]}>0,\right.$$
$$\left.\frac{\left(e^{DT}-1\right)\left[\gamma S_0\left(\mu_2 e^{-D\tau_2}-D\right)-K_2 D\left(1-e^{-DT}\right)\right]}{r_2\left[\gamma S_0+K_2\left(1-e^{-DT}\right)\right]}>0\right\} \tag{9}$$

where $\gamma = TD$.

Proof Let $(S(t), x_1(t), x_2(t), c(t))$ be any solution of system (2) with initial condition (3). From (8) or (9), we have

$$\frac{\mu_1 e^{-D\tau_1}\frac{\gamma S_0}{1-e^{-DT}}}{K_1+\frac{\gamma S_0}{1-e^{-DT}}} < D + r_1\frac{\gamma c_0 e^{-DT}}{1-e^{-DT}},$$
$$\frac{\mu_2 e^{-D\tau_2}\frac{\gamma S_0}{1-e^{-DT}}}{K_2+\frac{\gamma S_0}{1-e^{-DT}}} < D + r_2\frac{\gamma c_0 e^{-DT}}{1-e^{-DT}}.$$

Since $p(z) = \frac{\mu_i e^{-D\tau_i} z}{K_i+z}$, $i = 1, 2$ is strictly increasing for all $z \geq 0$, we may choose a sufficiently small positive constant ε such that

$$\frac{\mu_1 e^{-D\tau_1}\eta}{K_1+\eta} < D + r_1 m_4,$$
$$\frac{\mu_2 e^{-D\tau_2}\eta}{K_2+\eta} < D + r_2 m_4 \tag{10}$$

where

$$\eta = \frac{\gamma S_0}{1-e^{-DT}} + \varepsilon, \quad m_4 = \frac{\gamma c_0 e^{-DT}}{1-e^{-DT}} - \varepsilon.$$

It follows from that the first equation of system (2) that $S'(t) \leq -DS(t)$. So we consider the following impulse differential inequalities:

$$S'(t) \leq -DS(t), \quad t \neq nT, n \in N,$$
$$S(t^+) = S(t) + \gamma S_0, \quad t = nT,\ n \in N.$$

Using Lemma 2.1, we have

$$\limsup_{t\to\infty} S(t) \le \frac{\gamma S_0}{1-e^{-DT}}.$$

Hence, there exist a positive integer n_1 and an arbitrary small positive constant ε such that for all $t \ge n_1 T$,

$$S(t) \le \frac{\gamma S_0}{1-e^{-DT}} + \varepsilon =: \eta, \quad c(t) \ge \frac{\gamma c_0 e^{-DT}}{1-e^{-DT}} - \varepsilon = m_4. \tag{11}$$

From (11) and the second and the third equations of (2), we get that, for $t > n_1 T + \tau$,

$$\begin{aligned} x_1'(t) &\le \frac{\mu_1 \eta e^{-D\tau_1}}{K_1+\eta} x_1(t-\tau_1) - (D + r_1 m_4) x_1(t), \\ x_2'(t) &\le \frac{\mu_2 \eta e^{-D\tau_2}}{K_2+\eta} x_2(t-\tau_2) - (D + r_2 m_4) x_2(t). \end{aligned}$$

Consider the following comparison equation

$$\begin{aligned} z_1'(t) &= \frac{\mu_1 \eta e^{-D\tau_1}}{K_1+\eta} z_1(t-\tau_1) - (D + r_1 m_4) z_1(t), \\ z_2'(t) &= \frac{\mu_2 \eta e^{-D\tau_2}}{K_2+\eta} z_2(t-\tau_2) - (D + r_2 m_4) z_2(t). \end{aligned}$$

By Lemma 2.2 and (10), we obtain that

$$\lim_{t\to\infty} z_i(t) = 0, \quad i = 1, 2.$$

Since $x_i(s) = z_i(s) > 0, i = 1, 2$ for all $s \in [-\tau, 0]$, by the comparison theorem in differential equation and the nonnegativity of solution (with $x_i(t) \ge 0$), we have that $x_i(t) \to 0 (i = 1, 2)$ as $t \to \infty$. Without loss of generality, we may assume that $0 < x_i(t) < \varepsilon (i = 1, 2)$ for all $t \ge 0$, by the first equation of system (2), we have

$$S'(t) \ge -\left(D + \sum_{i=1}^{2} \frac{\mu_i \varepsilon}{\delta_i K_i}\right) S(t).$$

Consider the following impulse system

$$\begin{aligned} z_3'(t) &= -\left(D + \sum_{i=1}^{2} \frac{\mu_i \varepsilon}{\delta_i K_i}\right) z_3(t), \quad t \ne nT,\ n \in N, \\ z_3(t^+) &= z_3(t) + \gamma S_0, \quad t = nT,\ n \in N, \\ z_3(0^+) &= S(0^+). \end{aligned} \tag{12}$$

Then we have that, for $nT < t \leq (n+1)T$,

$$\widetilde{z_3}(t) = \frac{\gamma S_0 e^{-\left(D+\sum_{i=1}^{2}\frac{\mu_i \varepsilon}{\delta_i K_i}\right)(t-nT)}}{1-e^{-\left(D+\sum_{i=1}^{2}\frac{\mu_i \varepsilon}{\delta_i K_i}\right)T}}$$

is a unique globally asymptotically stable positive periodic solution of system (13). By Lemma 2.6, $\widetilde{z_3}(t) \leq S(t)$ and $\widetilde{z_3}(t) \to S^*(t)$ as $\varepsilon \to 0$. Hence, for any $\varepsilon_1 > 0$, there exists such a $T_1 > 0$ that, for $t > T_1$,

$$S(t) > \widetilde{z_3}(t) - \varepsilon_1. \tag{13}$$

On the other hand, from the first equation of (2), it follows that

$$S'(t) \leq -DS(t).$$

Consider the following comparison system

$$\begin{aligned} z_4'(t) &= -Dz_4(t), \quad t \neq nT,\ n \in N, \\ z_4(t^+) &= z_4(t) + \gamma S_0, \quad t = nT,\ n \in N, \\ z_4(0^+) &= S(0^+). \end{aligned} \tag{14}$$

Then we have

$$S(t) < \widetilde{z_4}(t) + \varepsilon_1 \tag{15}$$

as $t \to \infty$ and $\widetilde{z_4}(t) = S^*(t)$, where $\widetilde{z_4}(t)$ is a unique positive periodic solution of (15).

Let $\varepsilon \to 0$, then it follows from (13) and (15) that

$$S^*(t) - \varepsilon_1 < S(t) < S^*(t) + \varepsilon_1,$$

for t large enough, which implies $S(t) \to S^*(t)$ as $t \to \infty$.

Since the variables S, x_1 and x_2 do not appear in the fourth equation of system (2), then we only need to consider the subsystem of (2) as follows:

$$\begin{aligned} c'(t) &= -Dc(t), \quad t \neq nT,\ n \in N, \\ c(t^+) &= c(t) + \gamma c_0, \quad t = nT,\ n \in N, \\ c(0^+) &= c_{10} \geq 0. \end{aligned}$$

According to Lemma 2.4, we get that $c(t) \to c^*(t)$ as $t \to \infty$. This completes the proof.

Corollary 3.1 *Periodic solution* $(S^*(t), 0, 0, c^*(t))$ *of system* (2) *is globally attractive if*

$$\tau > \max\left\{\frac{1}{D}\ln\frac{\mu_1\gamma S_0\left(e^{DT}-1\right)}{\left[K_1\left(1-e^{-DT}\right)+\gamma S_0\right]\left[D\left(e^{DT}-1\right)+r_1\gamma c_0\right]},\right.$$
$$\left.\frac{1}{D}\ln\frac{\mu_2\gamma S_0\left(e^{DT}-1\right)}{\left[K_2\left(1-e^{-DT}\right)+\gamma S_0\right]\left[D\left(e^{DT}-1\right)+r_2\gamma c_0\right]}\right\}$$

where $\gamma = TD$.

Theorem 3.2 *There exist some constants* $m_2 > 0, m_3 > 0$ *such that* $\liminf_{t\to\infty} x_1 = m_2$ *and* $\liminf_{t\to\infty} x_2 = m_3$, *provided*

$$\gamma S_0 > \max\left\{\frac{K_1\left(D+\beta_1 L_2+r_1 M_4\right)\left(e^{(D+\mu_2 L_2/(\delta_2 K_2))T}-1\right)}{\mu_1 e^{-D\tau_1}-\left(D+\beta_1 L_2+r_1 M_4\right)} > 0,\right.$$
$$\left.\frac{K_2\left(D+\beta_2 L_1+r_2 M_4\right)\left(e^{(D+\mu_1 L_1/(\delta_1 K_1))T}-1\right)}{\mu_2 e^{-D\tau_2}-\left(D+\beta_2 L_1+r_2 M_4\right)} > 0\right\} \tag{16}$$

or

$$\gamma c_0 < \min\left\{\frac{\left(1-e^{-DT}\right)\left\{\gamma S_0\mu_1 e^{-D\tau_1}-\left[\gamma S_0+K_1\left(e^{(D+\mu_2 L_2/(\delta_2 K_2))T}-1\right)\right](D+\beta_1 L_2)\right\}}{r_1\left[\gamma S_0+K_1\left(e^{(D+\mu_2 L_2/(\delta_2 K_2))T}-1\right)\right]},\right.$$
$$\left.\frac{\left(1-e^{-DT}\right)\left\{\gamma S_0\mu_2 e^{-D\tau_2}-\left[\gamma S_0+K_2\left(e^{(D+\mu_1 L_1/(\delta_1 K_1))T}-1\right)\right](D+\beta_2 L_1)\right\}}{r_2\left[\gamma S_0+K_2\left(e^{(D+\mu_1 L_1/(\delta_1 K_1))T}-1\right)\right]}\right\}. \tag{17}$$

Proof Suppose that $(S(t), x_1(t), x_2(t), c(t))$ is any positive solution of system (2) with initial conditions (3). From the first equation of system (2), we have

$$\frac{dS}{dt} \geq -\left(D+\frac{\mu_1}{\delta_1 K_1}L_1+\frac{\mu_2}{\delta_2 K_2}L_2\right)S(t). \tag{18}$$

Consider the comparison system

$$\begin{aligned} z_5'(t) &= -\left(D+\frac{\mu_1}{\delta_1 K_1}L_1+\frac{\mu_2}{\delta_2 K_2}L_2\right)z_5(t), \quad t \neq nT,\\ z_5(t^+) &= z_5(t)+\gamma S_0, \quad t = nT,\\ z_5(0^+) &= S(0^+). \end{aligned} \tag{19}$$

Let $m_1 = \frac{\gamma S_0 \exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}L_1+\frac{\mu_2}{\delta_2 K_2}L_2\right)T\right]}{1-\exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}L_1+\frac{\mu_2}{\delta_2 K_2}L_2\right)T\right]} - \varepsilon > 0$. From Lemmas 2.1 and 2.6, we have $S(t) > m_1$ for t large enough.

From (16), we can choose ε small enough such that

$$\frac{\delta_1 K_1}{\mu_1}\left[\frac{1}{T}\ln\left(\frac{\gamma S_0\left[\mu_1 e^{-D\tau_1}-(D+\beta_1 L_2+r_1 M_4)\right]}{K_1\left(D+\beta_1 L_2+r_1 M_4\right)}+1\right)-\frac{\mu_2 L_2}{\delta_2 K_2}-D\right]>0,$$
$$\frac{\delta_2 K_2}{\mu_2}\left[\frac{1}{T}\ln\left(\frac{\gamma S_0\left[\mu_2 e^{-D\tau_2}-(D+\beta_2 L_1+r_2 M_4)\right]}{K_2\left(D+\beta_2 L_1+r_2 M_4\right)}+1\right)-\frac{\mu_1 L_1}{\delta_1 K_1}-D\right]>0.$$

Take

$$0<m_2<\frac{\delta_1 K_1}{\mu_1}\left[\frac{1}{T}\ln\left(\frac{\gamma S_0\left[\mu_1 e^{-D\tau_1}-(D+\beta_1 L_2+r_1 M_4)\right]}{K_1\left(D+\beta_1 L_2+r_1 M_4\right)}+1\right)-\frac{\mu_2 L_2}{\delta_2 K_2}-D\right],$$
$$0<m_3<\frac{\delta_2 K_2}{\mu_2}\left[\frac{1}{T}\ln\left(\frac{\gamma S_0\left[\mu_2 e^{-D\tau_2}-(D+\beta_2 L_1+r_2 M_4)\right]}{K_2\left(D+\beta_2 L_1+r_2 M_4\right)}+1\right)-\frac{\mu_1 L_1}{\delta_1 K_1}-D\right]. \tag{20}$$

In the following, we want to find $m_2>0$ and $m_3>0$, such that $x_1(t)>m_2$, $x_2(t)>m_3$ for t large enough. We will do it in the following two steps for convenience.

Step I: We will prove there exist $t_1, t_2 \in (0,\infty)$ such that $x_1(t_1)\geq m_2>0$ and $x_2(t_2)\geq m_3>0$. Otherwise, there will be three cases:

(i) There exists a $t_2>0$ such that $x_2(t_2)\geq m_3$, but $x_1(t)<m_2$ for all $t>0$;
(ii) There exists a $t_1>0$ such that $x_1(t_1)\geq m_2$, but $x_2(t)<m_3$ for all $t>0$;
(iii) $x_1(t)<m_2$, $x_2(t)<m_3$ for all $t>0$.

We first consider case (i). According to the above assumption, we get

$$\dot{S}(t)\geq-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)S(t). \tag{21}$$

By Lemma 2.6 on (21), we have $S(t)\geq z_6(t)$ and $z_6(t)\to\widetilde{z_6}(t)$ as $t\to\infty$, where $z_6(t)$ is the solution of

$$\begin{aligned} z_6'(t)&=-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)z_6(t), \quad t\neq nT,\\ z_6(t^+)&=z_6(t)+\gamma S_0, \quad t=nT,\\ z_6(0^+)&=S(0^+), \end{aligned} \tag{22}$$

and

$$\widetilde{z_6}(t)=\frac{\gamma S_0\exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)(t-nT)\right]}{1-\exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)T\right]}$$

$$\geq \frac{\gamma S_0 \exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)T\right]}{1-\exp\left[-\left(D+\frac{\mu_1}{\delta_1 K_1}m_2+\frac{\mu_2}{\delta_2 K_2}L_2\right)T\right]} =: \eta.$$

Then, from (20), we have

$$e^{-D\tau_1}\frac{\mu_1(\eta-\varepsilon)}{K_1+\eta-\varepsilon} > (\beta_1 L_2 + D + r_1 M_4). \tag{23}$$

Therefore, there exists a $\varepsilon > 0$ small enough, such that $S(t) \geq z_6(t) > \eta - \varepsilon$ and

$$\frac{dx_1(t)}{dt} \geq e^{-D\tau_1}\frac{\mu_1(\eta-\varepsilon)x_1(t-\tau_1)}{K_1+\eta-\varepsilon} - (\beta_1 L_2 + D + r_1 M_4)x_1(t). \tag{24}$$

By Lemma 2.2, (23) and (24), we have that $x_1(t) \to \infty$ as $t \to \infty$, which is a contradiction.

Similarly, we can prove $x_2(t) \to \infty$ as $t \to \infty$ in case (ii), which also is a contradiction.

Last, we consider case (iii). The second and the third equations of system (2) may be rewritten as follow:

$$\begin{aligned} x_1'(t) &= \left[\mu_1 e^{-D\tau_1}\frac{S(t)}{K_1+S(t)} - \beta_1 x_2(t) - (D + r_1 c(t))\right] \\ &\quad \times x_1(t) - \mu_1 e^{-D\tau_1}\frac{d}{dt}\int_{t-\tau_1}^{t}\frac{S(\theta)x_1(\theta)}{K_1+S(\theta)}d\theta, \\ x_2'(t) &= \left[\mu_2 e^{-D\tau_2}\frac{S(t)}{K_2+S(t)} - \beta_2 x_1(t) - (D + r_2 c(t))\right] \\ &\quad \times x_2(t) - \mu_2 e^{-D\tau_2}\frac{d}{dt}\int_{t-\tau_2}^{t}\frac{S(\theta)x_2(\theta)}{K_2+S(\theta)}d\theta. \end{aligned} \tag{25}$$

Define

$$V(t) = x_1(t) + x_2(t) + \mu_1 e^{-D\tau_1}\int_{t-\tau_1}^{t}\frac{S(\theta)x_1(\theta)}{K_1+S(\theta)}d\theta + \mu_2 e^{-D\tau_2}\int_{t-\tau_2}^{t}\frac{S(\theta)x_2(\theta)}{K_2+S(\theta)}d\theta.$$

Calculating the derivative of $V(t)$ along the solution of (2), it follows from (26) that

$$\begin{aligned} \frac{dV(t)}{dt} &= \left[\mu_1 e^{-D\tau_1}\frac{S(t)}{K_1+S(t)} - \beta_1 x_2(t) - (D + r_1 c(t))\right]x_1(t) \\ &\quad + \left[\mu_2 e^{-D\tau_2}\frac{S(t)}{K_2+S(t)} - \beta_2 x_1(t) - (D + r_2 c(t))\right]x_2(t). \end{aligned} \tag{26}$$

According to Lemma 2.5, for any $\varepsilon > 0$ small enough, there exists a positive constant T_1 such that for $t \geq T_1$,

$$c(t) \leq \frac{\gamma c_0}{1 - e^{-DT}} + \epsilon = M_4.$$

Hence follows from (26) that

$$\begin{aligned} \frac{dV(t)}{dt} &\geq \left[\mu_1 e^{-D\tau_1} \frac{S(t)}{K_1 + S(t)} - \beta_1 L_2 - (D + r_1 M_4)\right] x_1(t) \\ &\quad + \left[\mu_2 e^{-D\tau_2} \frac{S(t)}{K_2 + S(t)} - \beta_2 L_1 - (D + r_2 M_4)\right] x_2(t) \qquad (27) \\ &= (\beta_1 L_2 + (D + r_1 M_4)) \left[\frac{\mu_1 e^{-D\tau_1}}{\beta_1 L_2 + (D + r_1 M_4)} \cdot \frac{S(t)}{K_1 + S(t)} - 1\right] x_1(t) \\ &\quad + (\beta_2 L_1 + (D + r_2 M_4)) \left[\frac{\mu_2 e^{-D\tau_2}}{\beta_2 L_1 + (D + r_2 M_4)} \cdot \frac{S(t)}{K_2 + S(t)} - 1\right] x_2(t), \\ &\quad \text{for } t \geq T_1. \end{aligned}$$

From the first and fourth equations of system (2), we have

$$\begin{aligned} S'(t) &\geq -\left(D + \frac{\mu_1 m_2}{\delta_1 K_1} + \frac{\mu_2 L_2}{\delta_2 K_2}\right) S(t), \quad t \neq nT, \\ S(t^+) &= S(t) + \gamma S_0, \quad t = nT \end{aligned}$$

and

$$\begin{aligned} S'(t) &\geq -\left(D + \frac{\mu_1 L_1}{\delta_1 K_1} + \frac{\mu_2 m_3}{\delta_2 K_2}\right) S(t), \quad t \neq nT, \\ S(t^+) &= S(t) + \gamma S_0, \quad t = nT. \end{aligned}$$

Then, by Lemma 2.1, there exists such $T_2 \geq t_0 + \tau$, for $t \geq T_2$ that

$$S(t) > \frac{\gamma S_0 \exp\left[-\left(D + \frac{\mu_1}{\delta_1 K_1} m_2 + \frac{\mu_2}{\delta_2 K_2} L_2\right) T\right]}{1 - \exp\left[-\left(D + \frac{\mu_1}{\delta_1 K_1} m_2 + \frac{\mu_2}{\delta_2 K_2} L_2\right) T\right]} - \varepsilon =: \eta_1 \qquad (28)$$

and

$$S(t) > \frac{\gamma S_0 \exp\left[-\left(D + \frac{\mu_1}{\delta_1 K_1} L_1 + \frac{\mu_2}{\delta_2 K_2} m_3\right) T\right]}{1 - \exp\left[-\left(D + \frac{\mu_1}{\delta_1 K_1} L_1 + \frac{\mu_2}{\delta_2 K_2} m_3\right) T\right]} - \varepsilon =: \eta_2. \qquad (29)$$

From (20), (28), (29) and (16) or (17), we have

$$\frac{\mu_1 e^{-D\tau_1}}{\beta_1 L_2 + (D + r_1 M_4)} \cdot \frac{\eta_1}{K_1 + \eta_1} > 1 \tag{30}$$

and

$$\frac{\mu_2 e^{-D\tau_2}}{\beta_2 L_1 + (D + r_2 M_4)} \cdot \frac{\eta_2}{K_2 + \eta_2} > 1. \tag{31}$$

From (28), (28), (29), we have

$$\begin{aligned}\frac{dV(t)}{dt} &> (\beta_1 L_2 + (D + r_1 M_4))\left[\frac{\mu_1 e^{-D\tau_1}}{\beta_1 L_2 + (D + r_1 M_4)} \cdot \frac{\eta_1}{K_1 + \eta_1} - 1\right] x_1(t)\\ &\quad + (\beta_2 L_1 + (D + r_2 M_4))\left[\frac{\mu_2 e^{-D\tau_2}}{\beta_2 L_1 + (D + r_2 M_4)} \cdot \frac{\eta_2}{K_2 + \eta_2} - 1\right] x_2(t),\\ &\quad \text{for } t \geq T^* = \max\{T_1, T_2\}.\end{aligned} \tag{32}$$

Let

$$x_i^l = \min_{t \in [T^*, T^* + \tau]} x_i(t), \quad i = 1, 2.$$

We show that $x_i(t) \geq x_i^l$ for all $t \geq T^*$, $i = 1, 2$. Otherwise, there exists a nonnegative constant T_3 such that $x_i(t) \geq x_i^l$ for $t \in [T^*, T^* + \tau + T_3]$, $x(T^* + \tau + T_3) = x_i^l$ and $x_i'(T^* + \tau + T_3) \leq 0$. Thus from (30), (31) and the second and third equations of (2), we easily see that

$$\begin{aligned}x_i'(T^* + \tau + T_3) &> \left[\mu_i e^{-D\tau} \frac{\eta_i}{K_i + \eta_i} - (D + \beta_i L_j + r_i M_4)\right] x_i^l\\ &= (D + \beta_i L_j + r_i M_4)\left[\frac{\mu_i e^{-D\tau}}{D + \beta_i L_j + r_i M_4} \cdot \frac{\eta_i}{K_i + \eta_i} - 1\right] x_i^l\\ &> 0,\ i, j = 1, 2,\ i \neq j,\end{aligned}$$

which is a contradiction. Hence we get that $x_i(t) \geq x_i^l > 0$ for all $t \geq T^*$.

From (30), (31) and (32), we have that for $t \geq T^* = \max\{T_1, T_2\}$,

$$\begin{aligned}\frac{dV(t)}{dt} &> (\beta_1 L_2 + (D + r_1 M_4))\left[\frac{\mu_1 e^{-D\tau_1}}{\beta_1 L_2 + (D + r_1 M_4)} \cdot \frac{\eta_1}{K_1 + \eta_1} - 1\right] x_1^l\\ &\quad + (\beta_2 L_1 + (D + r_2 M_4))\left[\frac{\mu_2 e^{-D\tau_2}}{\beta_2 L_1 + (D + r_2 M_4)} \cdot \frac{\eta_2}{K_2 + \eta_2} - 1\right] x_2^l\\ &> 0,\end{aligned} \tag{33}$$

which implies $V(t) \to +\infty$ as $t \to +\infty$. This contradicts $V(t) \le \sum_{i=1}^{2}(1+\mu_i\tau_i e^{-D\tau_i})L_i$. Therefore, for any positive constant t_0, the inequality $x_1(t) < m_2$, $x_2(t) < m_3$ cannot hold for all $t \ge t_0$.

From the above three cases, we conclude that, there exist t_1, t_2 such that $x_1(t_1) \ge m_2, x_2(t_2) \ge m_3$.

Step II: If $x_i(t) \ge m_{i+1}, i = 1, 2$ for all t large enough, then our aim is obtained. Otherwise, $x_i(t)$ is oscillatory about m_{i+1}.

Let

$$m^*_{i+1} = \min\left\{\frac{m_{i+1}}{2}, m_{i+1}e^{-(D+\beta_i L_j + r_i M_4)\tau}\right\}, \quad i, j = 1, 2, \ j \ne i.$$

We claim $x_i(t) \ge m^*_{i+1}, i = 1, 2$. In the following, we shall prove that $x_i(t) \ge m^*_{i+1} (i = 1, 2)$. There exist positive constants $\bar{t}_i$ and ω_i such that

$$x_i(\bar{t}_i) = x_i(\bar{t}_i + \omega_i) = m_{i+1}$$

and

$$x_i(t) < m_{i+1}, \quad \text{for } \bar{t}_i < t < \bar{t}_i + \omega_i.$$

When $\bar{t}_i$ is large enough, $x_i(t) < m_{i+1}$ for $\bar{t}_i < t < \bar{t}_i + \omega_i$, then inequalities $S(t) > \eta_i$ and $c(t) \le M_4$ hold true. Since $x_i(t)$ is continuous and bounded and is not effected by impulses, we conclude that $x_i(t)$ is uniformly continuous. Hence there exists a constant T_4 (with $0 < T_4 < \tau$ and T_4 is independent of the choice of $\bar{t}_i$) such that $x_i(t) > \frac{m_{i+1}}{2}$ for all $\bar{t}_i \le t \le \bar{t}_i + T_4$. If $\omega_i \le T_4$, our aim is obtained. If $T_4 < \omega_i \le \tau$, from the second and the third equations of (2) we have that $x'_i(t) \ge -(D + \beta_i L_j + r_i M_4)x_i(t)$ for $\bar{t}_i < t \le \bar{t}_i + \omega_i, i, j = 1, 2, j \ne i$. Then we have $x_i(t) \ge m_{i+1}e^{-(D+\beta_i L_j + r_i M_4)\tau}$ for $\bar{t}_i < t \le \bar{t}_i + \omega_i \le \bar{t}_i + \tau$ since $x_i(\bar{t}_i) = m_{i+1}$. It is clear that $x_i(t) \ge m^*_{i+1}$ for $\bar{t}_i < t \le \bar{t}_i + \omega_i$. If $\omega_i \ge \tau$, then we have that $x_i(t) \ge m^*_{i+1}$ for $\bar{t}_i < t \le \bar{t}_i + \tau$. Thus, proceeding exactly as the proof for above claim, we can obtain $x_i(t) \ge m^*_{i+1}$ for $\bar{t}_i + \tau \le t \le \bar{t}_i + \omega_i$. Since the interval $[\bar{t}_i, \bar{t}_i + \omega_i]$ is arbitrarily chosen (we only need $\bar{t}_i$ to be large), we get that $x_i(t) \ge m^*_{i+1}$ for t large enough. In view of our arguments above, the choice of m^*_{i+1} is independent of the positive solution of (2) which satisfies that $x_i(t) \ge m^*_{i+1}$ for sufficiently large t. The proof is complete. □

From the above proof, we can obtain a corollary as follows:

Corollary 3.2 *The system (2) is permanent provided that (16) or (17) hold true.*

4 Discussion and numerical simulation

In this paper, we consider a Michaelis–Menten type competition chemostat model with impulsive input nutrient concentration and delayed growth response in a polluted environment. Our main aim is to investigate how the impulsive perturbation of the substrate, time delay for growth response and impulsive input toxicant affect the

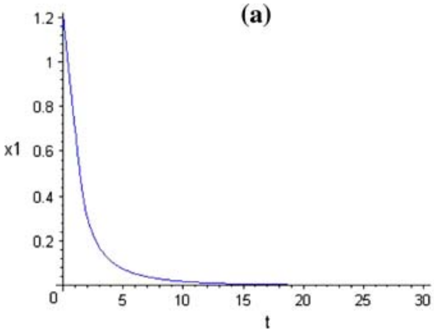

Fig. 1 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 0.5$, $\gamma c_0 = 0.3$, $T = 1$. **a** Time-series of $x_1(t)$. **b** Time-series of $x_2(t)$

dynamic behavior of the chemostat system. All these results show that dynamical behavior of system (2) becomes more complex under periodically impulsive inputting substrate. In Sect. 3, we give the conditions for the population of microorganisms will eventually be washed out of the chemostat and the conditions for the population of microorganisms will eventually be permanent. Theorem 3.1 shows if the impulsive periodic input nutrient concentration γS_0 is under certain value or the impulsive periodic input concentration the toxicant γc_0 is over certain value, then the population of microorganisms will be eventually extinct (see Fig. 1a, b). Then the microbial culture is failed. In this case, the substrate $S(t)$ and the microorganism $x_i(t)(i = 1, 2)$ can not coexist. In Sect. 3, we give the conditions for permanence of the microorganisms species. Theorem 3.2 shows if the impulsive periodic input concentration the nutrient γS_0 is over certain value or the impulsive periodic input concentration the toxicant γc_0 is under certain value, then the microorganisms species is permanent (see Figs. 2a, b; 3a, b). In this case, the microorganism is obtained. Then the microbial culture is successful. Obviously, if both the continuous culture and the impulsive culture can obtain the microorganism, the latter is better than the former since the impulsive cul-

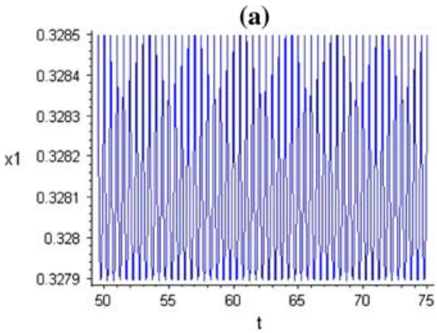

Fig. 2 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 1.2$, $T = 0.5$. **a** Time-series of $x_1(t)$ with $\gamma c_0 = 0.3$. **b** Time-series of $x_2(t)$ with $\gamma c_0 = 0.3$

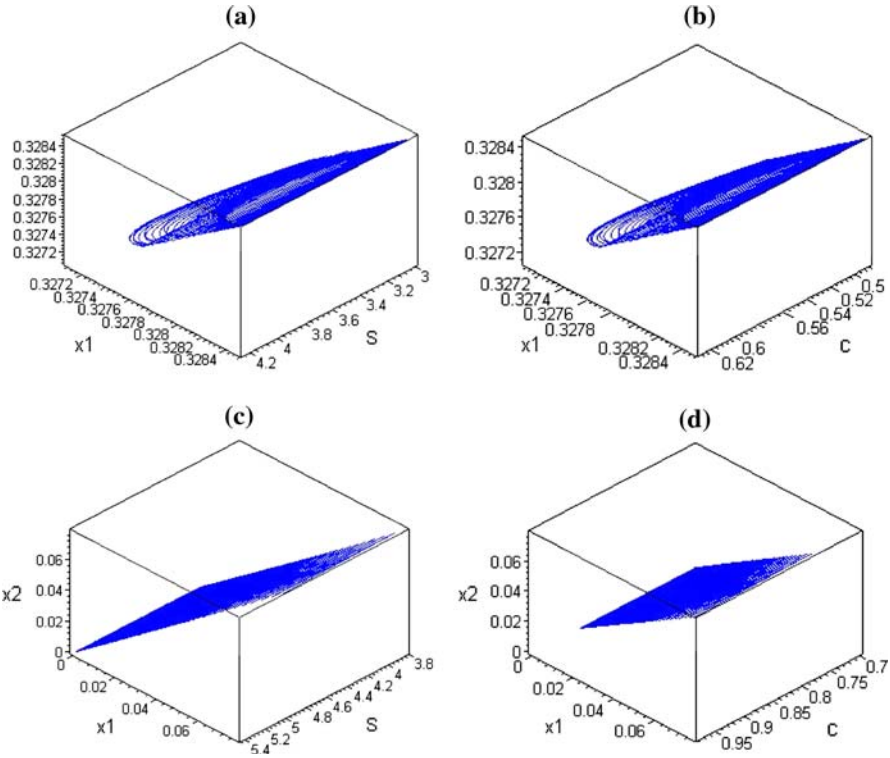

Fig. 3 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 1.2$, $T = 0.5$. **a** Phase portrait of $S(t)$, $x_1(t)$ and $x_2(t)$ with $\gamma c_0 = 0.3$. **b** Phase portrait of $c(t)$, $x_1(t)$ and $x_2(t)$ with $\gamma c_0 = 0.3$. **c** Phase portrait of $S(t)$, $x_1(t)$ and $x_2(t)$ with $\gamma c_0 = 0.9$. **d** Phase portrait of $c(t)$, $x_1(t)$ and $x_2(t)$ with $\gamma c_0 = 0.9$

ture can save the substrate. Whether the microorganism is extinct or not is determined completely by the input amount of the substrate γS_0 and concentration the toxicant γc_0 for the fixed period nT.

The environment with no pollution is in favor of living of the microorganism species. Otherwise, the polluted environment can lead to the microorganism species be extinct (see Fig. 3c, d). This shows that the input concentration of the toxicant greatly affects the dynamics behaviors of the model. We also note that the competition among the microorganism species can not lead them to be extinct.

From Corollary 3.1 and Theorem 3.2, we can see the extinction and permanence of the microorganism are dependent of time delays for growth response of the microorganism. Ultimately, when time delays for growth response is too long, the permanence of system disappears and the consumer population of the microorganism dies out, then we call it "profitless" time delays. This shows the sensitivity of the model dynamics on time delays (growth response). The ultimate scenario makes intuitive biological sense: if it takes too long to grow then the highest possible recruitment rate to the microorganism species ($\mu_i e^{-D\tau_i}(i = 1, 2)$) will drop below the losing rate to flow out D and the death of the microorganism population which is killed by the toxicant leading to the extinction of $x_i (i = 1, 2)$. This implies that time delays are significant influence on the dynamics behaviors of the model.

In conclusion, the fact that the microorganism cultures with variable yields exhibit sustained oscillations has an important implication for coexistence. In a sense, our results may provide a theoretical policy for the microorganism cultures in experiment.

In the following, we show the above results by numerical analysis. Then we consider the hypothetical set of parameter values as $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$.

If $\gamma S_0 = 0.5$, $\gamma c_0 = 0.3$, $T = 1$, then Theorem 3.1 hold true, which implies that the microorganism species be extinct (see Fig. 1a, b). Synchronously, the input concentration of the toxicant and the input concentration of the substrate exhibit periodic oscillation (see Fig. 4a, b). Figure 5a and b shows periodic solution $(S^*(t), 0, 0, c^*(t))$ of system (2) is globally attractive.

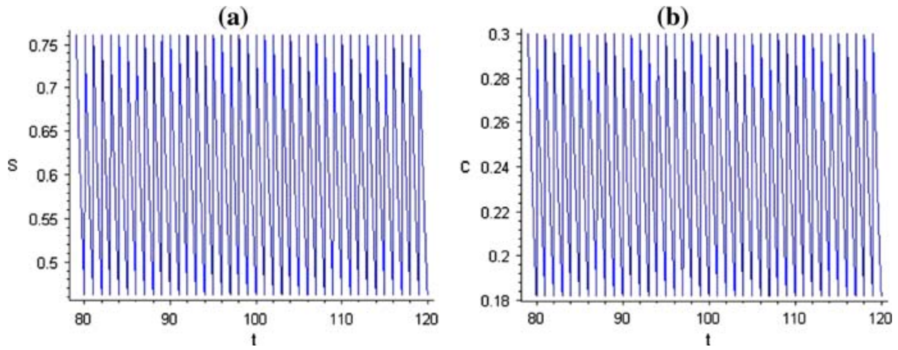

Fig. 4 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 0.5$, $\gamma c_0 = 0.3$, $T = 1$. **a** Time-series of $S(t)$. **b** Time-series of $c(t)$

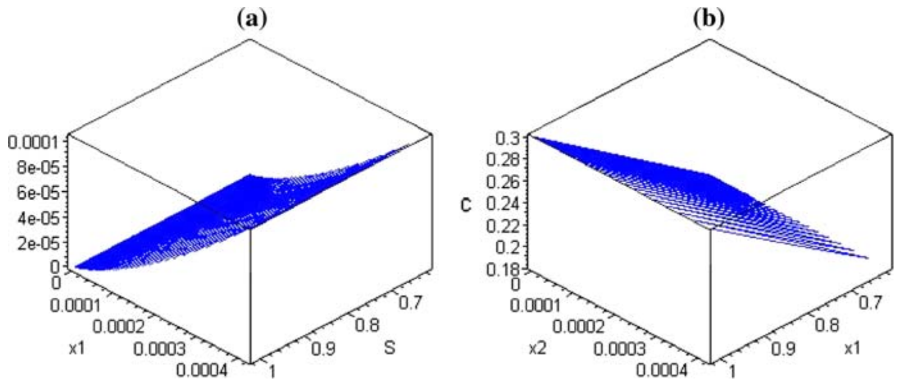

Fig. 5 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 0.5$, $\gamma c_0 = 0.3$, $T = 1$. **a** Phase portrait of $S(t)$, $x_1(t)$ and $x_2(t)$. **b** Phase portrait of $c(t)$, $x_1(t)$ and $x_2(t)$

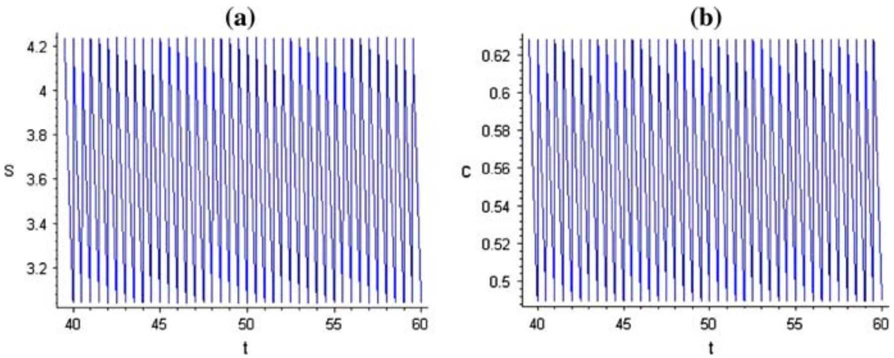

Fig. 6 Dynamical behavior of the system (2) with $\mu_1 = 1.1$, $\mu_2 = 1$, $\delta_1 = \delta_2 = 1$, $K_1 = 0.8$, $K_2 = 0.9$, $D = 0.5$, $r_1 = r_2 = 0.6$, $\beta_1 = \beta_2 = 0.2$, $\tau_1 = \tau_2 = 0.1$, $\gamma S_0 = 1.2$, $T = 0.5$. **a** Time-series of $S(t)$ with $\gamma c_0 = 0.3$. **b** Time-series of $c(t)$ with $\gamma c_0 = 0.3$

If $\gamma S_0 = 1.2$, $\gamma c_0 = 0.3$, $T = 0.5$, then Theorem 3.2 hold true, which implies that the microorganism species be permanent (see Figs. 2a, b; 3a, b). The input concentration of the toxicant and the input concentration of the substrate exhibit permanence and periodic oscillation (see Fig. 6a, b). Synchronously, that the microorganism population also exhibits periodic oscillation (see Fig. 2a, b). From Fig. 3a and b, we can see that system (2) has a globally asymptotically stable periodic solution.

If $\gamma c_0 = 0.9$ and the other parameter values are invariable, then the permanence of the microorganism species die out (see Fig. 3c, d). This shows that the input concentration of the toxicant can lead to the microorganism species be extinct.

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